

GLUING AFFINE TORUS ACTIONS VIA DIVISORIAL FANS

KLAUS ALTMANN, JÜRGEN HAUSEN, AND HENDRIK SÜSS

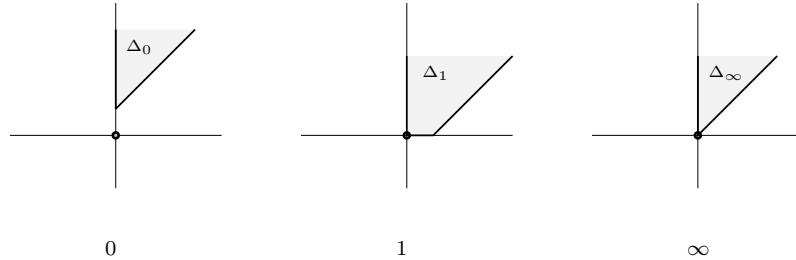
ABSTRACT. Generalizing the passage from a fan to a toric variety, we provide a combinatorial approach to construct arbitrary effective torus actions on normal, algebraic varieties. Based on the notion of a “proper polyhedral divisor” introduced in earlier work, we develop the concept of a “divisorial fan” and show that these objects encode the equivariant gluing of affine varieties with torus action. We characterize separateness and completeness of the resulting varieties in terms of divisorial fans, and we study examples like \mathbb{C}^* -surfaces and projectivizations of (non-split) vector bundles over toric varieties.

1. INTRODUCTION

This paper continues work of the first two authors [AH06], where the concept of “proper polyhedral divisors (pp-divisors)” was introduced in order to provide a complete description of normal affine varieties X that come with an effective action of an algebraic torus T . Recall that such a pp-divisor lives on a normal semiprojective (e.g., affine or projective) variety Y , and, at first glance, is just a finite linear combination

$$\mathcal{D} = \sum_D \Delta_D \otimes D,$$

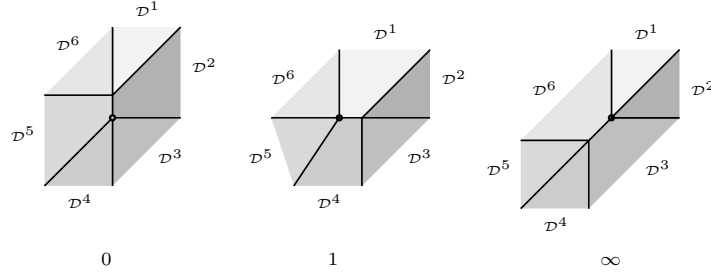
where D runs over the prime divisors of Y and the coefficients Δ_D are convex polyhedra, all living in a common rational vector space $N_{\mathbb{Q}}$ and all having the same pointed cone $\sigma \subseteq N_{\mathbb{Q}}$ as their tail. To see an example, let Y be the projective line, and take the points $0, 1$ and ∞ as prime divisors on Y . Then one obtains a pp-divisor \mathcal{D} on Y by prescribing polyhedral coefficients as follows.



The affine T -variety X associated to \mathcal{D} is the spectrum of a multigraded algebra A arising from \mathcal{D} . Evaluating the polyhedral coefficients turns the pp-divisor into a piecewise linear map from the dual cone $\sigma^\vee \subseteq M_{\mathbb{Q}}$ of the common tail to the rational Cartier divisors on Y : it sends $u \in \sigma^\vee$ to the divisor $\mathcal{D}(u) = \sum \alpha_D D$, where $\alpha_D = \min\langle u, \Delta_D \rangle$. The global sections of these evaluations fit together to the desired multigraded algebra:

$$A := \bigoplus_{u \in M \cap \sigma^\vee} \Gamma(Y, \mathcal{D}(u)).$$

In the present paper, we pass from the affine case to the general one. In the setting of toric varieties, the general case is obtained from the affine one by gluing cones to a fan. This is also our approach; applying Sumihiro's Theorem, we glue pp-divisors to a “divisorial fan”. There is an immediate naive idea of how such a divisorial fan should look: all its divisors \mathcal{D}^i live on the same semiprojective variety Y , their polyhedral coefficients Δ_D^i live in the same vector space $N_{\mathbb{Q}}$, and, for every prime divisor D , the Δ_D^i should form a polyhedral subdivision. For example, the single pp-divisor on $Y = \mathbb{P}^1$ discussed before could fit as a \mathcal{D}^1 into a divisorial fan comprising five further pp-divisors as indicated below.



To describe the gluing of affine T -varieties amounts to understanding their open subsets in terms of pp-divisors; a detailed study is given in Sections 3 and 4. Based on this, in Section 5 we introduce a concept of a divisorial fan. We show that each such divisorial fan canonically defines a normal variety with torus action (Theorem 5.3), and it turns out that every normal variety with effective torus action can be obtained in this way (Theorem 5.6). In Section 6, we discuss “coherent” divisorial fans—a special concept, which is much closer to the intuition than the general one. For example, the figure just drawn fits into this framework: it describes the projectivization of the cotangent bundle over the projective plane.

Following the philosophy of toric geometry that geometric properties of a toric variety should be read off from its defining combinatorial data, in Section 7 we study separateness and completeness and provide a complete characterization of these properties in terms of divisorial fans (Theorem 7.5). The last section is devoted to examples. We give the divisorial fans of Danilov-Gizatullin compactifications of affine \mathbb{K}^* -surfaces, recently discussed using different methods by Flenner, Kaliman and Zaidenberg. This example indicates that our constructions may be used for finding compactification of varieties with torus actions in a rather intuitive way. In our last example we provide a translation of Klyachko’s description of vector bundles on toric varieties into the picture of divisorial fans.

We expect applications of our constructions in all fields where toric varieties have proved their usefulness. Following the toric program, next steps will be the description of divisors, bundles and equivariant maps as well as the understanding of intersection products and cohomology on divisorial fans.

CONTENTS

1. Introduction	1
2. The affine case	3
3. Open embeddings	5
4. Patchworking	8
5. Divisorial Fans	11
6. Coherent fans	14
7. Separateness and Completeness	17
8. Further examples	21

2. THE AFFINE CASE

In this section, we briefly recall the basic concepts and results from [AH06], and we introduce some notions needed later. We begin with fixing our notation in convex geometry.

Throughout this paper, N denotes a lattice, i.e. a finitely generated free abelian group, and $M := \text{Hom}(N, \mathbb{Z})$ is the associated dual lattice. The rational vector space associated to N is $N_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} N$. Given a homomorphism $F: N \rightarrow N'$ of lattices, we write $F: N_{\mathbb{Q}} \rightarrow N'_{\mathbb{Q}}$ for the corresponding linear map. For two convex polyhedra $\Delta, \Delta' \subseteq N_{\mathbb{Q}}$, we write $\Delta \preceq \Delta'$ if Δ is a face of Δ' .

Let $\sigma \subseteq N_{\mathbb{Q}}$ be a pointed, convex, polyhedral cone. A σ -polyhedron is a convex polyhedron $\Delta \subseteq N_{\mathbb{Q}}$ having σ as its tail cone (also called recession cone). With respect to Minkowski addition, the set $\text{Pol}_{\sigma}^{+}(N)$ of all σ -polyhedra is a semigroup with cancellation law; we write $\text{Pol}_{\sigma}(N)$ for the associated Grothendieck group.

Now, let Y be a normal, algebraic variety defined over an algebraically closed field \mathbb{K} . Except in Section 7, we understand points always to be closed points of Y . The group of *polyhedral divisors* on Y is defined to be

$$\text{WDiv}_{\mathbb{Q}}(Y, \sigma) := \text{Pol}_{\sigma}(N) \otimes_{\mathbb{Z}} \text{WDiv}(Y),$$

where $\text{WDiv}_{\mathbb{Q}}(Y)$ denotes the group of rational Weil divisors on Y . Any polyhedral divisor $\mathcal{D} = \sum \Delta_D \otimes D$ with $\Delta_D \in \text{Pol}_{\sigma}^{+}(N)$ evaluates to a piecewise linear convex map on the dual cone $\sigma^{\vee} \subseteq M_{\mathbb{Q}}$ of $\sigma \subseteq N_{\mathbb{Q}}$, namely

$$\mathcal{D}: \sigma^{\vee} \rightarrow \text{WDiv}_{\mathbb{Q}}(Y), \quad u \mapsto \sum \text{eval}_u(\Delta_D) D, \quad \text{where } \text{eval}_u(\Delta_D) := \min_{v \in \Delta_D} \langle u, v \rangle.$$

Here, convexity is understood in the setting of divisors, this means that we always have $\mathcal{D}(u+u') \geq \mathcal{D}(u) + \mathcal{D}(u')$. A *proper polyhedral divisor* (abbreviated pp-divisor) is a polyhedral divisor $\mathcal{D} \in \text{WDiv}_{\mathbb{Q}}(Y, \sigma)$ such that

- (i) there is a representation $\mathcal{D} = \sum \Delta_D \otimes D$ with effective divisors $D \in \text{WDiv}_{\mathbb{Q}}(Y)$ and $\Delta_D \in \text{Pol}_{\sigma}^{+}(N)$,
- (ii) each evaluation $\mathcal{D}(u)$, where $u \in \sigma^{\vee}$, is a semiample \mathbb{Q} -Cartier divisor, i.e. has a base point free multiple,
- (iii) for any u in the relative interior of σ^{\vee} , some multiple of $\mathcal{D}(u)$ is a big divisor, i.e. admits a section with affine complement.

From now on, we suppose that Y is, additionally, semiprojective, i.e. projective over some affine variety. Every pp-divisor $\mathcal{D} = \sum \Delta_D \otimes D$ on Y defines a sheaf of graded \mathcal{O}_Y -algebras, and we have the corresponding relative spectrum:

$$\mathcal{A} := \bigoplus_{u \in M \cap \sigma^{\vee}} \mathcal{O}(\mathcal{D}(u)), \quad \tilde{X} := \text{Spec}_Y(\mathcal{A}).$$

The grading of \mathcal{A} gives rise to an effective action of the torus $T := \text{Spec}(\mathbb{K}[M])$ on \tilde{X} , the canonical map $\pi: \tilde{X} \rightarrow Y$ is a good quotient for this action, and for the field of invariant rational functions, we have

$$\mathbb{K}(\tilde{X})^T = \mathbb{K}(Y).$$

By [AH06, Theorem 3.1], the ring of global sections $A := \Gamma(\tilde{X}, \mathcal{O}) = \Gamma(Y, \mathcal{A})$ is finitely generated and normal, and there is a T -equivariant, projective, birational morphism $r: \tilde{X} \rightarrow X$ onto the normal, affine T -variety $X := X(\mathcal{D}) := \text{Spec}(A)$. Conversely, [AH06, Theorem 3.4] shows that every normal, affine variety with an effective torus action arises in this way.

The assignment from pp-divisors to normal, affine varieties with torus action is even functorial, see [AH06, Sec. 8]. Consider two pp-divisors,

$$\mathcal{D}' = \sum \Delta'_{D'} \otimes D' \in \text{PPDiv}_{\mathbb{Q}}(Y', \sigma'), \quad \mathcal{D} = \sum \Delta_D \otimes D \in \text{PPDiv}_{\mathbb{Q}}(Y, \sigma).$$

If $\psi: Y' \rightarrow Y$ is a morphism such that none of the supports of the D 's contains $\psi(Y)$, and if $F: N' \rightarrow N$ is a linear map with $F(\sigma') \subseteq \sigma$, then we set

$$\psi^*(\mathcal{D}) := \sum_D \Delta_D \otimes \psi^*(D), \quad F_*(\mathcal{D}') := \sum_{D'} (F(\Delta'_{D'}) + \sigma) \otimes D'.$$

Suppose that for some “polyhedral principal divisor” $\text{div}(\mathbf{f}) = \sum (v_i + \sigma) \otimes \text{div}(f_i)$ with $v_i \in N$ and $f_i \in \mathbb{K}(Y')$, we have inside $\text{WDiv}_{\mathbb{Q}}(Y', \sigma)$ the relation

$$\psi^*(\mathcal{D}) \leq F_*(\mathcal{D}') + \text{div}(\mathbf{f})$$

after evaluating with arbitrary $u \in \sigma^\vee$. Then the triple (ψ, F, \mathbf{f}) is called a map from the pp-divisor \mathcal{D}' to the pp-divisor \mathcal{D} . It induces homomorphisms of \mathcal{O}_Y -modules:

$$\mathcal{O}(\mathcal{D}(u)) \rightarrow \psi_* \mathcal{O}(\mathcal{D}'(F^*u)), \quad h \mapsto \mathbf{f}(u) \psi^*(h).$$

These maps fit together to a graded homomorphism $\mathcal{A} \rightarrow \psi_* \mathcal{A}'$. This in turn gives rise to a commutative diagram of equivariant morphisms, where the rows contain the geometric data associated to the pp-divisors \mathcal{D} and \mathcal{D}' respectively:

$$\begin{array}{ccccc} Y & \xleftarrow{\pi} & \tilde{X} & \xrightarrow{r} & X \\ \psi \uparrow & & \tilde{\varphi} \uparrow & & \uparrow \varphi \\ Y' & \xleftarrow{\pi'} & \tilde{X}' & \xrightarrow{r'} & X'. \end{array}$$

In particular, the map $\mathcal{D}' \rightarrow \mathcal{D}$ defines an equivariant morphism $X' \rightarrow X$ with respect to $T' \rightarrow T$ defined by $F: N' \rightarrow N$.

Here, we will frequently consider a special case of the above one. Namely, suppose that $Y' = Y$ and $\Delta'_D \subseteq \Delta_D$ holds for every prime divisor $D \in \text{WDiv}(Y)$. Then $\sigma^\vee \subseteq (\sigma')^\vee$ holds for the dualized tail cones. Moreover, for every $u \in \sigma^\vee$, we obtain

$$\mathcal{D}(u) = \sum_{v \in \Delta_D} \min \langle u, v \rangle D \leq \sum_{v \in \Delta'_D} \min \langle u, v \rangle D = \mathcal{D}'(u).$$

Consequently, we have a graded inclusion morphism $\mathcal{A} \hookrightarrow \mathcal{A}'$ of the associated sheaves of \mathcal{O}_Y -algebras, and hence a monomorphism $A \hookrightarrow A'$ on the level of global sections, which in turn determines a T -equivariant morphism $X' \rightarrow X$.

In [AH06, Prop. 7.8 and Cor. 7.9], we took a closer look at the fibers of the map $\pi: \tilde{X} \rightarrow Y$ arising from a pp-divisor $\mathcal{D} = \sum \Delta_D \otimes D$. Suppose that all D 's are prime. For a point $y \in Y$, its fiber polyhedron is the Minkowski sum

$$\Delta_y := \sum_{y \in D} \Delta_D \in \text{Pol}_\sigma^+(N).$$

Let Λ_y denote the normal fan of the fiber polyhedron Δ_y . Then Λ_y subdivides the cone σ^\vee , and the faces of Δ_y are in order reversing bijection to the cones of Λ_y via

$$F \mapsto \lambda(F) := \{u \in M_{\mathbb{Q}}; \langle u, v - v' \rangle \geq 0 \text{ for all } v \in \Delta, v' \in F\}.$$

Now, for $z \in \pi^{-1}(y)$, let $\omega(z)$ denote its orbit cone, i.e. the convex cone generated by all weights $u \in M$ admitting a u -homogeneous function on $\pi^{-1}(y)$ with $f(u) \neq 0$. Then there is a bijection:

$$\{T\text{-orbits in } \pi^{-1}(y)\} \rightarrow \Lambda_y \quad T \cdot \tilde{x} \mapsto \omega(\tilde{x}).$$

This does eventually provide an order and dimension preserving bijection between the T -orbits of $\pi^{-1}(y)$ and the faces of Δ_y .

Now, for the gluing of pp-divisors performed later, it is necessary to relax our notation: We will allow \emptyset as an element of $\text{Pol}_\sigma(N)$. This new element is subject to the rules $\emptyset + \Delta := \emptyset$ and $0 \cdot \emptyset := \sigma$. Moreover, if \emptyset occurs as a coefficient of a pp-divisor $\mathcal{D} = \sum \Delta_D \otimes D$, then we will always assume that $\bigcup_{\Delta_D = \emptyset} \text{supp } D$ is the support of an effective, semiample divisor, and we understand $\mathcal{D} \in \text{PPDiv}_\mathbb{Q}(Y, \sigma)$ as $\mathcal{D}|_{\text{Loc}(\mathcal{D})} \in \text{PPDiv}_\mathbb{Q}(\text{Loc}(\mathcal{D}), \sigma)$ with

$$\text{Loc}(\mathcal{D}) := Y \setminus \bigcup_{\Delta_D = \emptyset} \text{supp } D.$$

This new convention is compatible with the following evaluation of the coefficients of a polyhedral divisor.

Definition 2.1. Let N be a lattice, $\sigma \subseteq N_\mathbb{Q}$ a pointed polyhedral cone, and $\mathcal{D} = \sum \Delta_D \otimes D$ a polyhedral divisor on a normal variety Y . If

$$\mu: \{\text{prime divisors on } Y\} \rightarrow \mathbb{R}$$

is any map, then we define the associated *weighted sum of the polyhedral coefficients* to be

$$\Delta_\mu := \mathcal{D}_\mu := \mu(\mathcal{D}) := \sum \mu(D) \cdot \Delta_D \in \text{Pol}_\sigma(N).$$

Example 2.2. (i) For the trivial map $\mu \equiv 0$, the weighted sum Δ_0 gives the common tail cone $\text{tail}(\mathcal{D})$ of the coefficients of \mathcal{D} .

(ii) Fixing a prime divisor $P \in \text{WDiv}(Y)$, we may consider $\mu_P(D) := \delta_{D,P}$. The corresponding $\mathcal{D}_P = \mu_P(\mathcal{D})$ recovers the coefficient Δ_P of P .

(iii) Given a point $y \in Y$, set $\mu_y(D) := 1$ if $y \in D$ and $\mu_y(D) := 0$ else. Then $\mu_y(\mathcal{D})$ is precisely the fiber polyhedron Δ_y of the point $y \in Y$.

(iv) If $C \subseteq Y$ is a curve, then $\mu_C(D) := (C \cdot D)$ leads to $\mu_C(\mathcal{D}) =: (C \cdot \mathcal{D}) \in \text{Pol}_\sigma(N)$. In the case of $Y = C$, or $Y = \mathbb{P}^n$ and C being the line, we denote $(C \cdot \mathcal{D})$ also by $\deg \mathcal{D}$.

3. OPEN EMBEDDINGS

In this section, we begin the study of open embeddings of affine T -varieties in terms of pp-divisors. The first statement is a description of the equivariant basic open sets obtained by homogeneous localization. Recall that in toric geometry equivariant localization corresponds to passing to a face of a given cone. The generalization to pp-divisors involves also operations on the base variety; here is the precise procedure. Fix a lattice N and a normal, semiprojective variety Y . Moreover, let $\sigma \subseteq N_\mathbb{Q}$ be a pointed cone and consider a pp-divisor

$$\mathcal{D} = \sum \Delta_D \otimes D \in \text{PPDiv}_\mathbb{Q}(Y, \sigma).$$

As usual, \mathcal{A} denotes the associated sheaf of M -graded \mathcal{O}_Y -algebras, $A := \Gamma(Y, \mathcal{A})$ is the algebra of global sections, and we set $\tilde{X} := \text{Spec}_Y(\mathcal{A})$, and $X := \text{Spec}(A)$.

Definition 3.1. Let $w \in \sigma^\vee \cap M$ and $f \in A_w = \Gamma(Y, \mathcal{O}(\mathcal{D}(w)))$.

(i) The *face* of $\Delta \in \text{Pol}_\sigma^+(N)$ defined by w is

$$\text{face}(\Delta, w) := \{v \in \Delta; \langle w, v \rangle \leq \langle w, v' \rangle \text{ for all } v' \in \Delta\} \in \text{Pol}_{\sigma \cap w^\perp}^+(N).$$

(ii) The *zero set* of f and the *principal set* associated to f are

$$Z(f) := \text{Supp}(\text{div}(f) + \mathcal{D}(w)), \quad Y_f := Y \setminus Z(f).$$

(iii) The *localization* of the pp-divisor \mathcal{D} by f is

$$\mathcal{D}_f := \sum \text{face}(\Delta_D, w) \otimes D|_{Y_f} = \emptyset \otimes (\text{div}(f) + \mathcal{D}(w)) + \sum \text{face}(\Delta_D, w) \otimes D.$$

Lemma 3.2. *Let $w \in \sigma^\vee \cap M$ and $f \in A_w = \Gamma(Y, \mathcal{O}(\mathcal{D}(w)))$ as in Definition 3.1. Then, for $u \in \sigma \cap w^\vee$ and $k \gg 0$, one has $u + kw \in \sigma^\vee$ and*

$$\mathcal{D}_f(u) = \mathcal{D}(u + kw)|_{Y_f} - \mathcal{D}(kw)|_{Y_f}.$$

Proof. Set $\sigma_w := \sigma \cap w^\perp$ and $\Delta_w := \text{face}(\Delta, w)$. The first part of the assertion is clear by $\sigma_w = \sigma^\vee - \mathbb{Q}_{\geq 0}w$. The second part is obtained by comparing the non-empty coefficients of the prime divisors. For $\mathcal{D}_f(u)$, they are of the form $\min\langle \Delta_w, u \rangle$. If u attains this minimum at $v \in \Delta_w$, then v provides a minimal value for $u + kw$ on the whole Δ . Thus, the claim follows from

$$\min\langle \Delta_w, u \rangle = \langle v, u \rangle = \langle v, u + kw \rangle - \langle v, kw \rangle = \min\langle \Delta, u + kw \rangle - \min\langle \Delta, kw \rangle.$$

□

Proposition 3.3. *For a pp-divisor \mathcal{D} on a normal, semiprojective variety Y , let \mathcal{D}_f be the localization of \mathcal{D} by a homogeneous $f \in A_w$. Then \mathcal{D}_f is a pp-divisor on Y_f , and the canonical map $\mathcal{D}_f \rightarrow \mathcal{D}$ describes the open embedding $X_f \rightarrow X$.*

Proof. We may assume that \mathcal{D} has non-empty coefficients. Recall that Y_f is obtained by removing the support of $D = \text{div}(f) + \mathcal{D}(w)$ from Y . In particular, $\mathcal{D}(w)$ is principal on Y_f , and thus, for $k \gg 0$, Lemma 3.2 gives

$$\mathcal{O}_{Y_f}(\mathcal{D}_f(u)) \cong \mathcal{O}_{Y_f}(\mathcal{D}(u + kw)).$$

Using this, one sees that the assignment $u \mapsto \mathcal{D}_f(u)$ inherits from $u \mapsto \mathcal{D}(u)$ the properties (i) to (iii) of a pp-divisor formulated in Section 2.

To see that $\mathcal{D}_f \rightarrow \mathcal{D}$ defines an open embedding $X_f \rightarrow X$, it suffices to verify that, for any linear form $u \in (\sigma \cap w^\perp)^\vee \cap M = (\sigma^\vee \cap M) - \mathbb{N} \cdot w$, we have

$$\bigcup_{k \gg 0} \Gamma(Y, \mathcal{D}(u + kw)) / f^k = \Gamma(Y_f, \mathcal{D}(u + kw) - k\mathcal{D}(w)) \quad \text{where } k \gg 0.$$

Consider an element g/f^k of the left hand side. Then $\text{div}(g) + \mathcal{D}(u + kw) \geq 0$ holds. Hence, still on Y , we have

$$\text{div}(g/f^k) + \mathcal{D}(u + kw) - k\mathcal{D}(w) \geq -\text{div}(f^k) - k\mathcal{D}(w) = -kZ(f).$$

Thus, $\text{div}(g/f^k) + \mathcal{D}(u + kw) - k\mathcal{D}(w)$ is effective on Y_f , which means that g/f^k belongs to the right hand side.

For the reverse inclusion, take any element from the right hand side; we may write this element as g/f^k with $k \gg 0$. From the relation

$$\text{div}(g/f^k) + \mathcal{D}(u + kw) - k\mathcal{D}(w) \geq 0$$

on Y_f , we obtain the existence of an $\ell \in \mathbb{Z}$ such that the same divisor is $\geq -\ell Z(f)$ on Y . Moreover, we may assume that $\ell \geq k$. Then,

$$\text{div}(g/f^k) + \mathcal{D}(u + kw) - k\mathcal{D}(w) \geq -\text{div}(f^\ell) - \ell\mathcal{D}(w)$$

holds on Y . Using the convexity property of the assignment $u \mapsto \mathcal{D}(u)$, we can conclude

$$\text{div}(gf^{\ell-k}) + \mathcal{D}(u + \ell w) \geq \text{div}(gf^{\ell-k}) + \mathcal{D}(u + kw) + (\ell - k)\mathcal{D}(w) \geq 0.$$

However, this shows that $g/f^k = gf^{\ell-k}/f^\ell$ belongs to the big union of the left hand side. □

Whereas in toric geometry every equivariant open embedding of affine toric varieties is a localization, this needs no longer hold for general T -varieties. Thus, in view of equivariant gluing, we have to take care of more general affine open embeddings. We consider the following situation. By N , we denote again a lattice, and

Y is a normal variety. Moreover, $\sigma' \subseteq \sigma \subseteq N_{\mathbb{Q}}$ are pointed polyhedral cones, and we consider two pp-divisors

$$\mathcal{D}' = \sum \Delta'_D \otimes D \in \text{PPDiv}_{\mathbb{Q}}(Y, \sigma'), \quad \mathcal{D} = \sum \Delta_D \otimes D \in \text{PPDiv}_{\mathbb{Q}}(Y, \sigma).$$

We suppose that $\Delta'_D \subseteq \Delta_D$ holds for every prime divisor $D \in \text{WDiv}(Y)$. For the respective loci of these divisors, we then obtain

$$V' := \text{Loc}(\mathcal{D}') = \{y \in Y; \Delta'_y \neq \emptyset\} \subseteq \{y \in Y; \Delta_y \neq \emptyset\} = \text{Loc}(\mathcal{D}) =: V.$$

Note that we have a natural map $\mathcal{D}' \rightarrow \mathcal{D}$ of pp-divisors. As mentioned in Section 2, this gives rise to a commutative diagram of T -equivariant morphisms, where the rows contain the geometric data associated to \mathcal{D} and \mathcal{D}' respectively:

$$\begin{array}{ccccc} V & \xleftarrow{\pi} & \tilde{X} & \xrightarrow{r} & X \\ \uparrow & & \uparrow & & \uparrow \\ V' & \xleftarrow{\pi'} & \tilde{X}' & \xrightarrow{r'} & X' \end{array}$$

Proposition 3.4. *The morphism $X' \rightarrow X$ associated to $\mathcal{D}' \rightarrow \mathcal{D}$ is an open embedding if and only if any $y \in V'$ admits $w \in \sigma^\vee \cap M$ and $f \in A_w$ with*

$$y \in V_f \subseteq V', \quad \Delta'_y = \text{face}(\Delta_y, w), \quad \text{face}(\Delta'_v, w) = \text{face}(\Delta_v, w) \text{ for every } v \in V_f.$$

Proof. Suppose that $X' \rightarrow X$ is an open embedding. For short we write $X' \subseteq X$. Given $y \in V'$, let $T \cdot z' \subseteq (\pi')^{-1}(y)$ the (unique) closed T -orbit and choose $f \in A_w$, where $w \in \sigma^\vee \cap M$, such that

$$f(r'(z')) \neq 0, \quad f|_{X \setminus X'} = 0.$$

Then we always have $X_f = X'_f$. Since the maps $r: \tilde{X} \rightarrow X$ and $r': \tilde{X}' \rightarrow X'$ are birational and proper, this implies

$$B := \Gamma(\tilde{X}_f, \mathcal{O}) = \Gamma(X_f, \mathcal{O}) = \Gamma(X'_f, \mathcal{O}) = \Gamma(\tilde{X}'_f, \mathcal{O}) =: B'.$$

Considering the invariant parts, we obtain that $\Gamma(V_f, \mathcal{O})$ equals $\Gamma(V'_f, \mathcal{O})$. Since both $V'_f \subseteq V_f$ are semiprojective, this gives $V'_f = V_f$. Using [AH06, Thm. 3.1 (iii)], we arrive at

$$y \in \pi(r^{-1}(X_f)) = V_f = V'_f \subseteq V'.$$

Moreover, B and B' are the algebras of global sections of the localized pp-divisors \mathcal{D}_f and \mathcal{D}'_f living on $V_f = V'_f$. By [AH06, Lemma. 9.1], $B = B'$ implies $\mathcal{D}_f = \mathcal{D}'_f$. Thus, we obtain

$$\text{face}(\Delta'_v, w) = \text{face}(\Delta_v, w)$$

for every $v \in V_f$. Finally, $f(r'(z')) \neq 0$ implies $\Delta'_y = \text{face}(\Delta'_y, w)$, which, together with the preceding observation, shows $\Delta'_y = \text{face}(\Delta_y, w)$.

Now suppose that \mathcal{D}' and \mathcal{D} satisfy the assumptions of the proposition. For every $y \in V'$ choose w and $f \in A_w$ as in the assertion. Then we have $\Delta'_y = \text{face}(\Delta'_y, w)$. From this, we can conclude $(\pi')^{-1}(y) \subseteq \tilde{X}'_f$. Consequently, the sets \tilde{X}'_f , where $y \in V'$, cover \tilde{X}' .

Moreover, the assumption implies that the localized pp-divisors \mathcal{D}'_f and \mathcal{D}_f coincide. Hence, the canonical maps $\tilde{X}'_f \rightarrow \tilde{X}_f$ are isomorphisms, and this also holds for the canonical maps $X'_f \rightarrow X_f$. Since \tilde{X}' is covered by the sets \tilde{X}'_f , we obtain that $X' \rightarrow X$ is an open embedding. \square

Remark 3.5. Suppose we are in the situation of Proposition 3.4.

- (i) The condition $\text{face}(\Delta'_v, w) = \text{face}(\Delta_v, w)$ for every $v \in V_f$ is equivalent to the following one: If $\text{face}(\Delta'_D, w) \neq \text{face}(\Delta_D, w)$, then $D \in \text{WDiv}(Y)$ is a prime divisor supported in $Z(f)$.
- (ii) The condition of the previous Proposition 3.4 implies that $\Delta'_y \preceq \Delta_y$ holds for all $y \in Y$. This weaker condition turns out to be equivalent to the map $\tilde{X}' \rightarrow \tilde{X}$ being an open embedding.

Example 3.6. Let $Y = \mathbb{P}^1$ and $N = \mathbb{Z}$. The pp-divisor $\mathcal{D} = [0, \infty) \otimes \{0\} + [1, \infty) \otimes \{\infty\}$ describes \mathbb{K}^2 with its standard \mathbb{K}^* -action. On the other hand, we may consider $\mathcal{D}' := [0, \infty) \otimes \{0\} + \emptyset \otimes \{\infty\}$. The morphism $\mathcal{D}' \rightarrow \mathcal{D}$ describes the blowing up of the origin in \mathbb{K}^2 , hence, it is not an open embedding.

4. PATCHWORKING

In this section, we continue the study of equivariant open embeddings. Given a pp-divisor and its associated affine T -variety X , our aim is to construct a pp-divisor for an invariant, affine, open subset $X' \subseteq X$. Clearly, X' is a union of homogeneous localizations of X . We need the following setting.

Definition 4.1. Let X be an affine T -variety, and let $X' \subseteq X$ be an invariant, open, affine subset. We say that $f_1, \dots, f_r \in \Gamma(X, \mathcal{O})$ *reduce* X to X' if

- (i) each f_i is homogeneous and $X' = \bigcup_{i=1}^r X_{f_i}$ holds,
- (ii) each f_i is invertible on some orbit closure in X' .

Remark 4.2. For any invariant, affine, open subset $X' \subseteq X$ of an affine T -variety X , there exist homogeneous functions $f_1, \dots, f_r \in \Gamma(X, \mathcal{O})$ that reduce X to X' . If $X' \hookrightarrow X$ is an open embedding that arises from a map of pp-divisors $\mathcal{D}' \rightarrow \mathcal{D}$ as in Proposition 3.4, then the functions $f \in A_w$ mentioned there will do.

By [AH06, Thm. 8.8], the open embedding $X' \hookrightarrow X$ may be represented by some map of pp-divisors $\mathcal{D}' \rightarrow \mathcal{D}$. In the following, we will show that \mathcal{D}' may be chosen to live on the same base Y as \mathcal{D} does.

Proposition 4.3. Consider a pp-divisor $\mathcal{D} = \sum_D \Delta_D \otimes D$ on a normal semiprojective variety Y , denote the associated geometric data by

$$Y \xleftarrow{\pi} \tilde{X} \xrightarrow{r} X,$$

and let $X' \subseteq X$ be an invariant, affine, open subset. Then $Y' := \pi(r^{-1}(X')) \subseteq Y$ is open and semiprojective. Moreover, if $f_i \in A_{w_i}$ reduce X to X' , then

$$\mathcal{D}' := \bigcup \mathcal{D}_{f_i} := \sum \Delta'_D \otimes D|_{Y'}, \quad \text{where } \Delta'_D := \bigcup_{D \cap Y'_{f_i} \neq \emptyset} \text{face}(\Delta_D, w_i) \preceq \Delta_D$$

is a pp-divisor on $Y' = \bigcup Y_{f_i}$, and the canonical map $\mathcal{D}' \rightarrow \mathcal{D}$ defines an open embedding of affine varieties having X' as its image.

Corollary 4.4. Let X be the affine variety arising from a pp-divisor \mathcal{D} on a normal variety Y , and let the pp-divisors

$$\mathcal{D}' = \sum \Delta'_D \otimes D = \bigcup \mathcal{D}_{f_i}, \quad \mathcal{D}'' = \sum \Delta''_D \otimes D = \bigcup \mathcal{D}_{g_j}$$

with loci $Y' \subseteq Y$ and $Y'' \subseteq Y$, respectively, describe open subsets $X' = \bigcup X_{f_i}$ and $X'' = \bigcup X_{g_j}$ as in Proposition 4.3. Then we have

$$\mathcal{D}' \cap \mathcal{D}'' := \sum (\Delta'_D \cap \Delta''_D) \otimes D = \bigcup \mathcal{D}_{f_i g_j}.$$

In particular, $\mathcal{D}' \cap \mathcal{D}''$ is a pp-divisor with locus $Y' \cap Y'' \subseteq Y$, and the canonical map $\mathcal{D}' \cap \mathcal{D}'' \rightarrow \mathcal{D}$ describes an open embedding having $X' \cap X''$ as its image.

For the proof of Proposition 4.3, we need two preparatory lemmas. Let Y, Y'' be normal semiprojective varieties, N, N'' lattices, $\sigma \subset N_{\mathbb{Q}}$ and $\sigma'' \subset N''_{\mathbb{Q}}$ pointed cones, and consider pp-divisors

$$\mathcal{D} = \sum \Delta_D \otimes D \in \text{PPDiv}_{\mathbb{Q}}(Y, \sigma), \quad \mathcal{D}'' = \sum \Delta_{D''} \otimes D'' \in \text{PPDiv}_{\mathbb{Q}}(Y'', \sigma'')$$

with non-empty coefficients. Moreover, let (ψ, F, f) be a map from \mathcal{D}'' to \mathcal{D} . As indicated in Section 2, the map (ψ, F, f) gives rise to a commutative diagram of equivariant morphisms, where the rows contain the geometric data associated to \mathcal{D} and \mathcal{D}'' respectively:

$$\begin{array}{ccccc} Y & \xleftarrow{\pi} & \tilde{X} & \xrightarrow{r} & X \\ \psi \uparrow & & \tilde{\varphi} \uparrow & & \uparrow \varphi \\ Y'' & \xleftarrow{\pi''} & \tilde{X}'' & \xrightarrow{r''} & X'' \end{array}$$

Lemma 4.5. *In the above notation, suppose that the morphism $\varphi: X'' \rightarrow X$ is an open embedding. Then the following holds.*

- (i) *We have $\tilde{\varphi}(\tilde{X}'') = r^{-1}(\varphi(X''))$, and the induced morphism $\tilde{\varphi}: \tilde{X}'' \rightarrow \tilde{\varphi}(\tilde{X}'')$ is proper and birational.*
- (ii) *The image $\psi(Y'') \subseteq Y$ is open and semiprojective, and $\psi: Y'' \rightarrow \psi(Y'')$ is a projective birational morphism.*
- (iii) *For every $y \in \psi(Y'')$, the intersection $\tilde{U}_y := \pi^{-1}(y) \cap \tilde{\varphi}(\tilde{X}'')$ contains a unique T -orbit that is closed in \tilde{U}_y .*

Proof. Consider the open subset $U := \varphi(X'')$, its inverse image $\tilde{U} := r^{-1}(U)$ and $V := \pi(\tilde{U})$. By [Ha05, Lemma 2.1], the latter set is open in Y . These data fit into the commutative diagram

$$\begin{array}{ccccc} V & \xleftarrow{\pi} & \tilde{U} & \xrightarrow{r} & U \\ \psi \uparrow & & \tilde{\varphi} \uparrow & & \cong \uparrow \varphi \\ Y'' & \xleftarrow{\pi''} & \tilde{X}'' & \xrightarrow{r''} & X'' \end{array}$$

The map $\tilde{\varphi}$ is birational, because r'' , φ and r are birational. Moreover, since r'' and hence $\varphi \circ r''$ are proper, we infer from the diagram that $\tilde{\varphi}$ is proper, and thus surjective. Consequently, we obtain $\psi(Y'') = V$; in particular, this set is open in Y . In order to see that V is a semiprojective variety, note first that for its global functions, we have

$$\Gamma(V, \mathcal{O}) \cong \Gamma(\tilde{U}, \mathcal{O})_0 \cong \Gamma(\tilde{X}'', \mathcal{O})_0 \cong \Gamma(Y'', \mathcal{O});$$

the first equality is guaranteed by [Ha05, Lemma 2.1]. Thus, setting $A_0'' := \Gamma(Y'', \mathcal{O})$ and $Y_0'' := \text{Spec}(A_0'')$, we obtain a commutative diagram

$$\begin{array}{ccc} Y'' & \xrightarrow{\psi} & V \\ & \searrow & \swarrow \\ & Y_0'' & \end{array}$$

Since ψ is surjective, $V \rightarrow Y_0''$ is proper. Since V is quasiprojective, we even obtain that $V \rightarrow Y_0''$ is projective, and so is $\psi: Y'' \rightarrow V$. The map ψ is also birational,

because we have the commutative diagram

$$\begin{array}{ccc} \mathbb{K}(Y) & \xrightarrow{\psi^*} & \mathbb{K}(Y'') \\ \downarrow = & & \downarrow = \\ \mathbb{K}(\tilde{X})_0 & \xrightarrow[\tilde{\varphi}^*]{\cong} & \mathbb{K}(\tilde{X}'')_0 \end{array}$$

Now, consider $y \in V$ and the intersection $\tilde{U}_y := \pi^{-1}(y) \cap \tilde{\varphi}(\tilde{X}'')$. Since $\pi^{-1}(y)$ contains only finitely many T -orbits, the same holds for \tilde{U}_y . Let $T \cdot z_1, \dots, T \cdot z_r$ be the closed T -orbits of \tilde{U}_y . We claim

$$\psi^{-1}(y) = \pi''(\tilde{\varphi}^{-1}(\tilde{U}_y)) = \bigcup_{i=1}^r \pi''(\tilde{\varphi}^{-1}(T \cdot z_i)).$$

The first equality is clear by surjectivity of $\tilde{\varphi}$ and the quotient maps π, π'' . The second one is verified below; it uses properness of $\tilde{\varphi}$: Given $y'' \in \pi''(\tilde{\varphi}^{-1}(\tilde{U}_y))$, we have $y'' = \pi''(z'')$ for some $z'' \in \tilde{\varphi}^{-1}(\tilde{U}_y)$. Since π'' is constant on orbit closures, we may assume that $T'' \cdot z''$ is closed in $\tilde{\varphi}^{-1}(\tilde{U}_y)$. By properness of $\tilde{\varphi}$, the image $\tilde{\varphi}(T'' \cdot z'') = T \cdot \tilde{\varphi}(z'')$ is closed in \tilde{U}_y . It follows that y'' belongs to the right hand side.

Having verified the claim, we may proceed as follows. The closed invariant subsets $\tilde{\varphi}^{-1}(T \cdot z_i) \subseteq \tilde{X}''$ are pairwise disjoint. By the properties of the good quotient $\pi'': \tilde{X}'' \rightarrow Y''$, the images $\pi''(\tilde{\varphi}^{-1}(T \cdot z_i))$ are pairwise disjoint as well. In particular, $\psi^{-1}(y)$ is disconnected if $r > 1$. The latter is impossible because ψ , as a birational projective morphism between normal varieties, has connected fibers. \square

Lemma 4.6. *For the functions $f_i \in A_{w_i}$ of Proposition 4.3, we always have*

$$\Delta'_D := \bigcup_{D \cap Y'_{f_i} \neq \emptyset} \text{face}(\Delta_D, w_i) \preceq \Delta_D.$$

In particular, for every f_i with $D \cap Y'_{f_i} \neq \emptyset$, we have $\text{face}(\Delta_D, w_i) \preceq \Delta'_D$.

Proof. Let D be a prime divisor intersecting Y' , and consider a point $y \in D \cap Y'$ such that $y \in Y_{f_i}$ holds for all the f_i of Proposition 4.3 with $D \cap Y_{f_i} \neq \emptyset$ and D is the only prime divisor with $\Delta_D \neq 0$ containing y . Then we have

$$\Delta_y = \Delta_D.$$

According to [AH06, Thm. 8.8], the inclusion $X' \subseteq X$ is described by a map of pp-divisors. Thus, we may apply Lemma 4.5 and obtain that there is a unique closed T -orbit $T \cdot z$ in $\pi^{-1}(y) \cap r^{-1}(X')$. This orbit corresponds to a face $F(z) \preceq \Delta_y$ via

$$T \cdot z \mapsto \omega(z) \mapsto \text{face}(\Delta_y, u) \text{ with } u \in \text{int } \omega(z),$$

where $\text{int } \omega(z)$ denotes the relative interior of the cone $\omega(z)$. Since $z \in r^{-1}(X')$ holds, some of the f_i of Proposition 4.3 satisfies $f_i(z) \neq 0$ but vanishes on $\overline{T \cdot z} \setminus T \cdot z$, where the closure is taken in $\pi^{-1}(y)$. This means $w_i \in \text{int } \omega(z)$. To conclude the proof, it suffices to show

$$\bigcup_{D \cap Y'_{f_j} \neq \emptyset} \text{face}(\Delta_y, w_j) = \text{face}(\Delta_y, w_i).$$

For any f_j with $D \cap Y'_{f_j} \neq \emptyset$, we have $y \in Y_{f_j}$. Thus, there is a point $z_j \in \pi^{-1}(y) \cap X'$ with $f_j(z_j) \neq 0$. We may choose z_j such that $\omega(z_j)$ is minimal; this means $w_j \in \text{int } \omega(z_j)$. Since $z \in \overline{T \cdot z_j}$ holds, we obtain $\omega(z) \preceq \omega(z_j)$. This in turn implies $\text{face}(\Delta_y, w_j) \preceq \text{face}(\Delta_y, w_i)$, and the above equation follows. \square

Proof of Proposition 4.3. By [AH06, Thm. 8.8], there is a map (ψ, F, f) of pp-divisors $\mathcal{D}'' \rightarrow \mathcal{D}$ such that the associated morphism is the open embedding of X' into X . Lemma 4.5 gives $Y' = \psi(Y'')$, and it ensures openness and semiprojectivity of $Y' \subseteq Y$. Lemma 4.6 tells us

$$\Delta'_D := \bigcup_{i=1}^r \text{face}(\Delta_D, w_i) \preceq \Delta_D.$$

Property 4.1 (ii) ensures that any X_{f_i} contains the generic orbit closure of X' . Consequently, all X_{f_i} have the same weight cone. Hence, the tail cone of $\text{face}(\Delta_D, w_i)$ does not depend on i , and, thus, \mathcal{D}' is a well defined polyhedral divisor on Y' .

In order to verify the pp-properties for \mathcal{D}' , we only have to concern ourselves with semiample and bigotry. Locally, we have canonical isomorphisms

$$\psi^*(\mathcal{D}') = \psi^*(\mathcal{D}'_{f_i}) = \psi^*(\mathcal{D}_{f_i}) = \mathcal{D}''_{\psi^* f_i} = \mathcal{D}''.$$

This shows $\psi^*(\mathcal{D}') \cong \mathcal{D}''$, and thus we obtain both properties by pushing forward suitable global sections.

Finally, the fact that the map $\mathcal{D}' \rightarrow \mathcal{D}$ is an open embedding with image X' follows by comparing the induced maps $\mathcal{D}'_{f_i} \rightarrow \mathcal{D}_{f_i}$ of the localizations. \square

5. DIVISORIAL FANS

In toric geometry, the equivariant gluing of affine pieces is described by means of a fan, i.e. a collection of polyhedral cones satisfying natural compatibility conditions. In this section, we generalize this idea and present a natural concept to describe the equivariant gluing of affine varieties with torus action.

Definition 5.1. Let N be a lattice, $\sigma', \sigma \subseteq N_{\mathbb{Q}}$ pointed cones, Y a normal, semiprojective variety, and consider two pp-divisors on Y :

$$\mathcal{D}' = \sum \Delta'_D \otimes D \in \text{PPDiv}_{\mathbb{Q}}(Y, \sigma'), \quad \mathcal{D} = \sum \Delta_D \otimes D \in \text{PPDiv}_{\mathbb{Q}}(Y, \sigma).$$

We call \mathcal{D}' a *face* of \mathcal{D} (written $\mathcal{D}' \preceq \mathcal{D}$) if $\Delta'_D \subseteq \Delta_D$ holds for all D and for any $y \in \text{Loc}(\mathcal{D}')$ there are $w_y \in \sigma^\vee \cap M$ and a D_y in the linear system $|\mathcal{D}(w_y)|$ with

- (i) $y \notin \text{supp}(D_y)$,
- (ii) $\Delta'_y = \text{face}(\Delta_y, w_y)$,
- (iii) $\text{face}(\Delta'_v, w_y) = \text{face}(\Delta_v, w_y)$ for every $v \in Y \setminus \text{supp}(D_y)$.

By Proposition 3.4, the face relation “ $\mathcal{D}' \preceq \mathcal{D}$ ” is the combinatorial counterpart of the open embeddings after applying the functor $X(\bullet)$. It implies that the tail cone of \mathcal{D}' is a face of the tail cone of \mathcal{D} . This enables us to generalize the concept of a fan to the setting of pp-divisors. Recall that in the preceding section, we introduced the intersection of two polyhedral divisors $\mathcal{D}' = \sum \Delta'_D \otimes D$ and $\mathcal{D} = \sum \Delta_D \otimes D$ on a common variety Y as

$$\mathcal{D}' \cap \mathcal{D} := \sum (\Delta'_D \cap \Delta_D) \otimes D.$$

Definition 5.2. Let N be a lattice, and Y a normal, semiprojective variety. A *divisorial fan* on (Y, N) is a set \mathcal{S} of pp-divisors $\mathcal{D} \in \text{PPDiv}_{\mathbb{Q}}(Y, \sigma_{\mathcal{D}})$ with tail cones $\sigma_{\mathcal{D}} \subseteq N_{\mathbb{Q}}$ such that for any two $\mathcal{D}', \mathcal{D} \in \mathcal{S}$ the intersection $\mathcal{D}' \cap \mathcal{D}$ is a face of both \mathcal{D}' and \mathcal{D} and, moreover, belongs to \mathcal{S} .

Given a divisorial fan $\mathcal{S} = \{\mathcal{D}^i; i \in I\}$ on a normal variety Y , we have the affine T -varieties $X_i := X(\mathcal{D}^i)$, and for any two $i, j \in I$, the T -equivariant open embeddings

$$X_i \xleftarrow{\eta_{ij}} X(\mathcal{D}^i \cap \mathcal{D}^j) \xrightarrow{\eta_{ji}} X_j.$$

We denote the associated images by $X_{ij} := \eta_{ij}(X(\mathcal{D}^i \cap \mathcal{D}^j)) \subseteq X_i$. Then we have T -equivariant isomorphisms $\varphi_{ij} := \eta_{ji} \circ \eta_{ij}^{-1}$ from X_{ij} onto X_{ji} .

Theorem 5.3. *The affine T -varieties X_i and the isomorphisms $\varphi_{ij}: X_{ij} \rightarrow X_{ji}$ are gluing data. The resulting space*

$$X := X(\mathcal{S}) := \bigsqcup X_i / \sim, \quad \text{where } X_{ij} \ni x \sim \varphi_{ij}(x) \in X_{ji},$$

is a prevariety with affine diagonal $X \rightarrow X \times X$, and it comes with a (unique) T -action such that all canonical maps $X_i \rightarrow X$ are equivariant.

Proof. The only thing to check is that the maps $\varphi_{ij}: X_{ij} \rightarrow X_{ji}$ do in fact define gluing data. Concretely, this means to verify two things, namely

$$\varphi_{ij}(X_{ij} \cap X_{ik}) = X_{ji} \cap X_{jk} \quad \text{and} \quad \varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}.$$

The first one of these identities can be directly deduced from the following observation: for any triple $i, j, k \in I$, there is a commutative diagram

$$\begin{array}{ccccc} X(\mathcal{D}^i \cap \mathcal{D}^j) & \xrightarrow{\eta_{ij}} & X(\mathcal{D}^i) & \xleftarrow{\eta_{ik}} & X(\mathcal{D}^i \cap \mathcal{D}^k) \\ & \swarrow & \uparrow & \searrow & \\ & & X(\mathcal{D}^i \cap \mathcal{D}^j \cap \mathcal{D}^k) & & \end{array}$$

and Corollary 4.4 yields that $X(\mathcal{D}^i \cap \mathcal{D}^j \cap \mathcal{D}^k)$ is mapped onto the intersection $X_{ij} \cap X_{ik}$ of $X_{ij} = \eta_{ij}(X(\mathcal{D}^i \cap \mathcal{D}^j))$ and $X_{ik} = \eta_{ik}(X(\mathcal{D}^i \cap \mathcal{D}^k))$.

The second identity then may be verified as an identity of rational maps: all X_i have the same function field, and the pull back maps φ_{ij}^* are the identity. \square

Example 5.4. The figure in the introduction on Page 2 shows a divisorial fan on the projective line \mathbb{P}^1 generated by six maximal pp-divisors $\mathcal{D}^1, \dots, \mathcal{D}^6$, all of them of the form

$$\mathcal{D}^i = \Delta_0^i \otimes \{0\} + \Delta_1^i \otimes \{1\} + \Delta_\infty^i \otimes \{\infty\}.$$

In order to indicate in the figure that a polyhedron is a coefficient of the divisor \mathcal{D}^i , we put the label “ \mathcal{D}^i ” on it. As we will see in Section 8, the corresponding T -variety is the projectivized cotangent bundle $\mathbb{P}(\Omega_{\mathbb{P}^2})$ over the projective plane.

We conclude this section with a recipe for producing lots of examples using toric geometry. Firstly we recall from [AH06] a toric construction of pp-divisors.

Let N' be a lattice, and denote by $M' := \text{Hom}(N', \mathbb{Z})$ the dual lattice. Let $\delta \subseteq N'_{\mathbb{Q}}$ be a pointed polyhedral cone. We consider the associated affine toric variety and its big torus

$$X' := \mathbb{T}\mathbb{V}(\delta) := \text{Spec } \mathbb{K}[\delta^\vee \cap M'], \quad T' := \text{Spec } \mathbb{K}[M'].$$

Suppose that $T \rightarrow T'$ is a monomorphism of tori arising from a surjection $\text{deg}: M' \rightarrow M$ of the respective (character) lattices; in the case of $N' = \mathbb{Z}^n$ and $\delta = \mathbb{Q}_{\geq 0}^n$, this map just fixes multidegrees $\text{deg}(z_i) \in M$ for the coordinates z_1, \dots, z_n of $X^T = \mathbb{K}^n$.

The aim is to construct a pp-divisor for the induced T -action on X' . We will work in terms of the following exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M'' & \xrightarrow{p^*} & M' & \xrightarrow{\text{deg}} & M \longrightarrow 0 \\ & & & \searrow t^* & \swarrow s & & \\ 0 & \longleftarrow & N'' & \xleftarrow{p} & N' & \xleftarrow{\text{deg}^*} & N \longleftarrow 0 \\ & & & \searrow t & \swarrow s^* & & \end{array}$$

Here we have, additionally, chosen a section s of $\text{deg}: M' \rightarrow M$; this corresponds to a section t of $p: N' \rightarrow N''$ via $t^* = \text{id}_{M'} - s \circ \text{deg}$.

Let Σ'' be the fan in $N_{\mathbb{Q}}''$ that is obtained as the coarsest common subdivision of the images of the faces of δ under p ; its support is $|\Sigma''| = p(\delta)$. Similarly, we obtain a subdivision of $\omega := \deg(\delta^\vee)$ inside $M_{\mathbb{Q}}$. Then the *positive fiber* of $u \in \omega \cap M$ is

$$\Delta(u) := (\deg^{-1}(u) \cap \delta^\vee) - s(u) = t^*(\deg^{-1}(u) \cap \delta^\vee) \subseteq M_{\mathbb{Q}}''.$$

The normal fans $\Lambda(\Delta(u))$ of these polytopes vary with the chamber structure of ω . Their coarsest common refinement is exactly the fan Σ'' .

The toric variety $Y'' = \mathbb{T}\mathbb{V}(\Sigma'')$ associated to Σ'' is the Chow quotient of X' by the action of T . The polytopes $\Delta(u)$ correspond to semiample divisors on Y'' , and they are precisely the evaluations of the pp-divisor on Y' describing X' as a T -variety, namely

$$\mathcal{D}' := \sum_{\varrho \in (\Sigma'')^{(1)}} \Delta_{\varrho} \otimes D_{\varrho},$$

where, by abuse of notation, the ray $\varrho \in (\Sigma'')^{(1)}$ is identified with its primitive lattice vector $\varrho \in N''$, where D_{ϱ} denotes the invariant prime divisor corresponding to ϱ , and, dualizing the previous formula for $\Delta(u)$,

$$\Delta_{\varrho} := \Delta_{\varrho}(\delta) := (p^{-1}(\varrho) \cap \delta) - t(\varrho) = s^*(p^{-1}(\varrho) \cap \delta) \subseteq N_{\mathbb{Q}}.$$

Now, more generally, let X' be a semiprojective, not necessarily affine, toric variety arising from a fan Σ' in N' . Then, similarly to the above setup, we may consider the coarsest common refinement Σ'' of all images $p(\delta)$ where $\delta \in \Sigma'$. This is again a fan in $N_{\mathbb{Q}}''$, and we denote by Y'' the associated toric variety.

In addition, we have, for every cone $\delta \in \Sigma'$, the previous construction $\Sigma''(\delta)$ refining all projected faces of δ . Note that each $|\Sigma''(\delta)|$ is a union of cones of Σ'' and we may define a polyhedral divisor on Y'' by

$$\mathcal{D}'(\delta) := \sum_{\varrho \in (\Sigma'')^{(1)}} \Delta_{\varrho}(\delta) \otimes D_{\varrho}.$$

This pp-divisor is the pullback of the one previously associated to the affine chart $X(\delta) \subseteq X'$. Its locus equals $\mathbb{T}\mathbb{V}(\Sigma'' \cap |\Sigma''(\delta)|)$ which is a modification of $\mathbb{T}\mathbb{V}(\Sigma''(\delta))$. Elements $\varrho \in (\Sigma'')^{(1)} \setminus |\Sigma''(\delta)|$ lead to $\Delta_{\varrho}(\delta) = \emptyset$ in a natural way. The pp-divisors $\mathcal{D}'(\delta)$, where $\delta \in \Sigma'$, obviously fit together to a divisorial fan \mathcal{S}' on Y' , and this divisorial fan describes the T -action on the toric variety X' .

Proposition 5.5. *Let $X \subseteq X'$ be a closed T -invariant subvariety with $X \cap T' \neq \emptyset$, and let $\iota: Y \rightarrow Y''$ be the normalization of the closure of the image of $X \cap T'$ in Y'' . Then the $\mathcal{D}(\delta) := \iota^*(\mathcal{D}'(\delta))$ fit together to a divisorial fan $\mathcal{S} := \iota^*\mathcal{S}'$ on Y , and this divisorial fan describes the T -variety X .*

We leave the proof of this observation to the reader. Note that if $X \cap T'$ is given by T -homogeneous equations $f_i \in \Gamma(T', \mathcal{O})$, where $i \in I$, then, multiplying with $\chi^{-s(\deg f_i)}$ shifts them into $\mathbb{K}[M'']$, and we obtain Y as the normalized closure in Y'' of

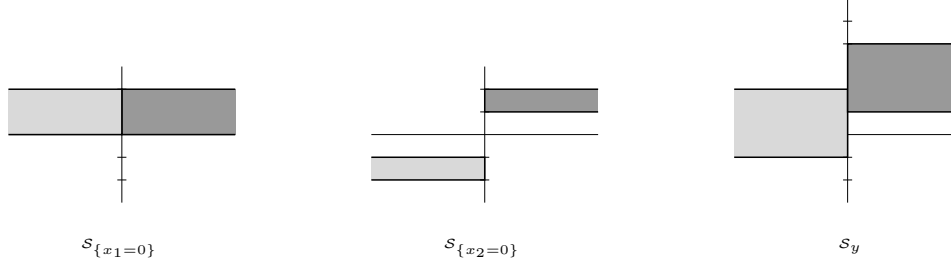
$$V(\chi^{-s(\deg f_i)} f_i; i \in I) \subseteq T'' = \text{Spec } \mathbb{K}[M''].$$

The previous proposition presents a kind of an algorithm for constructing a divisorial fan for T -varieties that are equivariantly embedded in a toric variety, e.g. in a projective space. Besides this, we have the following.

Theorem 5.6. *Up to equivariant isomorphism, every normal variety with an effective algebraic torus action arises from a divisorial fan.*

The next example shows that the slices \mathcal{S}_y , in contrast to the special case \mathcal{S}_P , need not be polyhedral subdivisions.

Example 6.4. Let $Y = \mathbb{A}_{\mathbb{K}}^2$ and denote by y the origin. Then, Δ_y^1 and Δ_y^2 are the two polyhedra in the rightmost figure, but $\Delta_y^1 \cap \Delta_y^2$ is not a face of the Δ_y^i . However, \mathcal{S}_y is still a complex since $\Delta_y^{12} = \emptyset$.



Definition 6.5. A set of pp-divisors $\mathcal{S} = \{\mathcal{D}^i = \sum_D \Delta_D^i \otimes D\}$ will be called *coherent* if for any i, j there is a $u^{ij} \in M$ such that for all D there is a c_D^{ij} with

$$\max\langle \Delta_D^i, u^{ij} \rangle \leq c_D^{ij} \leq \min\langle \Delta_D^j, u^{ij} \rangle$$

and

$$\Delta_D^i \cap [\langle \bullet, u^{ij} \rangle = c_D^{ij}] = \Delta_D^j \cap [\langle \bullet, u^{ij} \rangle = c_D^{ij}].$$

In principle, coherence means that the coefficients of any two polyhedral divisors \mathcal{D}^i and \mathcal{D}^j are separated by hyperplanes which are mutually parallel in all prime divisor slices \mathcal{S}_D ; see the figure shown in the introduction. However, we do not exclude the case of $u^{ij} = 0$. Then coherence means that $\Delta_D^i = \Delta_D^j$ whenever both polytopes are non-empty.

The divisorial fans we obtained in Proposition 5.5 are coherent. Here, we would like to raise the opposite question: If \mathcal{S} is a set of pp-divisors, then we denote by $\langle \mathcal{S} \rangle$ the set of all intersections of elements of \mathcal{S} . If \mathcal{S} is coherent, under which conditions does $\langle \mathcal{S} \rangle$ become a divisorial fan?

Proposition 6.6. Let $\mathcal{S} = \{\mathcal{D}^i\}$ be a coherent set of pp-divisors. Then $\langle \mathcal{S} \rangle$ inherits the coherence as well as the pp-property. Moreover, for any non-negative map $\mu: \{\text{prime divisors on } Y\} \rightarrow \mathbb{R}_{\geq 0}$, the slices $\langle \mathcal{S} \rangle_\mu$ are polyhedral subdivisions of $N_{\mathbb{Q}}$.

Proof. First, it is easy to show that coherence survives under finite intersections: Checking this comes down to considering elements $\mathcal{D}^i, \mathcal{D}^j, \mathcal{D}^k \in \mathcal{S}$ and trying to separate $\mathcal{D}^i \cap \mathcal{D}^j$ from \mathcal{D}^k . If, say, $k = j$, then this is done by u^{ij} . On the other hand, if $p \notin \{i, j\}$, then one takes $u^{ik} + u^{jk}$ instead.

Now, we will see why the pp-property remains valid for intersections $\mathcal{D}^i \cap \mathcal{D}^j$ of \mathcal{S} -elements. Denoting $u := u^{ij}$ from the definition of coherence and $Z^{i/j} := \bigcup_{\Delta_D^{i/j} \neq \emptyset} D$, the loci of $\mathcal{D}^{i/j}$ are the semiprojective $Y^{i/j} := Y \setminus Z^{i/j}$. Since both Z^i and Z^j are supports of effective, semiample divisors, we can use the sum of these divisors to show that $Z^i \cup Z^j$ is of the same quality, i.e. $Y^i \cap Y^j \subseteq Y$ stays semiprojective. Moreover, the locus Y^{ij} of $\mathcal{D}^i \cap \mathcal{D}^j$ is

$$\begin{aligned} Y^{ij} &= (Y^i \cap Y^j) \setminus \bigcup_{\Delta_D^i \cap \Delta_D^j \neq \emptyset} D = (Y^i \cap Y^j) \setminus \text{supp} \sum_D (\min\langle \Delta_D^j, u \rangle - \max\langle \Delta_D^i, u \rangle) D \\ &= (Y^i \cap Y^j) \setminus \text{supp} (\mathcal{D}^i(-u) + \mathcal{D}^j(u)). \end{aligned}$$

From $-u \in \text{tail}(\mathcal{D}^i)^\vee$ and $u \in \text{tail}(\mathcal{D}^j)^\vee$, we obtain that the divisors $\mathcal{D}^i(-u)$ and $\mathcal{D}^j(u)$ are semiample on Y^i and Y^j , respectively. Hence their sum is semiample on $Y^i \cap Y^j$. Since $\mathcal{D}^i(-u) + \mathcal{D}^j(u)$ is also effective, this shows the semiprojectivity of Y^{ij} .

We may exploit another fact from this: The previous equation shows that $\mathcal{D}^i(-u) =$

$-\mathcal{D}^j(u)$ on Y^{ij} . Thus, on the locus of $\mathcal{D}^i \cap \mathcal{D}^j$, the divisor $-\mathcal{D}^j(u)$ is semiample. On the other hand, denoting $\mathcal{D}^j = \sum_D \Delta_D^j \otimes D$, Lemma 3.2 tells us that $\mathcal{D}' := \sum_D \text{face}(\Delta_D^j, u) \otimes D$ leads to the evaluations $\mathcal{D}'(u') = \mathcal{D}(u' + \ell u)|_{Y^{ij}} - \mathcal{D}(\ell u)|_{Y^{ij}}$ for $\ell \gg 0$. Hence, they are semiample, too.

Eventually, we may assume $\mathcal{S} = \langle \mathcal{S} \rangle$ to deal with the polyhedral subdivision \mathcal{S}_μ . Under summation according to μ , the coherent separation of the polyhedra inside the coefficients \mathcal{S}_P transfers to \mathcal{S}_μ as $\max\langle \Delta_\mu^i, u^{ij} \rangle < \min\langle \Delta_\mu^j, u^{ij} \rangle$ or $\max\langle \Delta_\mu^i, u^{ij} \rangle = \min\langle \Delta_\mu^j, u^{ij} \rangle =: c_\mu^{ij}$ with $\Delta_\mu^i \cap [\langle \bullet, u^{ij} \rangle = c_\mu^{ij}] = \Delta_\mu^j \cap [\langle \bullet, u^{ij} \rangle = c_\mu^{ij}]$. \square

The main ingredient of divisorial fans is the face relation, i.e. the issue of open embeddings. It is one of the advantages of coherent pp-sets that the class of open embeddings among its elements is easier to describe than in the general case.

Lemma 6.7. *Let \mathcal{D} be a pp-divisor on Y with non-empty coefficients, let $Z \subseteq Y$ be the support of an effective, semiample divisor. For $u \in \text{tail}(\mathcal{D})^\vee$, we define \mathcal{D}' on $Y \setminus Z$ via the u -faces of the \mathcal{D} -coefficients. If \mathcal{D}' is assumed to be pp (i.e. if $-\mathcal{D}(u)$ is semiample on $Y \setminus Z$), then $X(\mathcal{D}') \rightarrow X(\mathcal{D})$ is an open embedding if and only if*

$$\forall y \in Y \setminus Z \quad \exists D_y \in |\mathcal{D}(\mathbb{N} \cdot u)| : y \notin \text{supp } D_y =: Z_y \supseteq Z.$$

Proof. By definition of \mathcal{D}' , we know that $\Delta'_y = \text{face}(\Delta_y, u)$, hence $\text{face}(\Delta'_y, u) = \text{face}(\Delta_y, u)$ for all points $y \in Y \setminus Z$. In particular, the condition in the lemma is stronger than that of Proposition 3.4 or Definition 5.1, hence it is sufficient for having an open embedding.

Out of necessity, let us assume that \mathcal{D}' is a face of \mathcal{D} in the sense of Definition 5.1. Then, for any $y \in \text{Loc}(\mathcal{D}') = Y \setminus Z$, there are $u_y \in \sigma^\vee \cap M$ and $D_y \in |\mathcal{D}(u_y)|$ with $y \notin \text{supp } D_y \supseteq Z$ and $\Delta'_y = \text{face}(\Delta_y, u_y)$. Since u defines the same face of Δ_y , both u_y and u are contained in the interior of the normal cone $\mathcal{N}(\Delta'_y, \Delta_y)$. Thus, we can find another $u' \in \text{int } \mathcal{N}(\Delta'_y, \Delta_y)$ such that $u_y + u' = k \cdot u$ for some $k \gg 0$. The semiamplessness of $\mathcal{D}(u')$ provides some $E \in |\mathcal{D}(u')|$ avoiding y , and we may use $D_y + E \in |\mathcal{D}(u_y) + \mathcal{D}(u')| = |\mathcal{D}(ku)|$ as the new divisor D_y . \square

The openness condition of Lemma 6.7 looks like asking for semiamplessness of some divisor. This is indeed the case if $Z = \emptyset$. In the general case, however, we can only formulate a sufficient condition in these terms: The condition of Lemma 6.7 is fulfilled whenever there is an effective, semiample divisor E with $\text{supp } E = Z$ and $(k \mathcal{D}(u) - E)$ being semiample for $k \gg 0$. A very special example for this situation is when $E \sim \mathcal{D}(u)$ – this is what happens in the case of the localizations described in Proposition 3.3. A second class of easy instances is, of course, when Y is affine. Summarizing our considerations so far, we obtain

Corollary 6.8. *Let $\mathcal{S} = \{\mathcal{D}^j\}$ be a coherent set of pp-divisors such that, for any i, j , there is an effective, semiample divisor E^{ij} on $\text{Loc}(\mathcal{D}^j) \subseteq Y$ with $\text{supp } E^{ij} = \bigcup \{D \mid \Delta_D^i \cap \Delta_D^j = \emptyset\}$ and $k \mathcal{D}^j(u^{ij}) - E^{ij}$ being semiample for $k \gg 0$. Then $\langle \mathcal{S} \rangle$ is a divisorial fan.*

Proof. Corollary 4.4 establishes the relation between intersection of pp-divisors, i.e. of their polyhedral coefficients, and the intersection of the corresponding affine T -varieties. Hence, the claim follows from Proposition 6.6 and, via the previous remarks, from Lemma 6.7. \square

We will conclude this section by taking a closer look at the situation of two special cases. If Y is either a smooth, projective curve (covering the one-codimensional torus actions) or $Y = \mathbb{P}^n$, then everything becomes very clear – and we even have a straight characterization of the pp-ness of the elements of \mathcal{S} :

Proposition 6.9. *Let $\mathcal{S} = \{\mathcal{D}^j\}$ be a coherent set of polyhedral divisors on a projective Y with either $\dim Y = 1$ or $Y = \mathbb{P}^n$. Then, the elements \mathcal{D}^j are pp-divisors if and only if $\deg \mathcal{D}^j \subsetneq \text{tail } \mathcal{D}^j$ and (being automatically satisfied if Y is rational) $\mathcal{D}^j(u) \sim 0$ for any $u \in (\text{tail } \mathcal{D}^j)^\vee$ with $\min\langle \deg \mathcal{D}^j, u \rangle = 0$. Assuming this, $\langle \mathcal{S} \rangle$ becomes a divisorial fan if and only if $\min\langle \deg \mathcal{D}^j, u^{ij} \rangle = 0$ implies that $\deg \mathcal{D}^i \cap \deg \mathcal{D}^j \neq \emptyset$.*

Proof. First, in case of $\deg \mathcal{D}^j = \emptyset$, all conditions mentioned in the proposition are automatically fulfilled. On the other hand, this case means that $\text{Loc}(\mathcal{D}^j) \subsetneq Y$, i.e. that $\text{Loc}(\mathcal{D}^j)$ is affine. In particular, all conditions characterizing pp-properties or openness of maps among the corresponding affine T -varieties are satisfied, too. Thus, we may assume that $\deg \mathcal{D}^j \neq \emptyset$. In the proposition, the condition on the single polyhedral divisors \mathcal{D}^j translates into $\deg \mathcal{D}^j(u) > 0$, i.e. ampleness, or $\mathcal{D}^j(u) \sim 0$, where the latter cannot occur for $u \in \text{int}(\text{tail } \mathcal{D}^j)^\vee$. This does exactly characterize pp-ness. Now, assuming that \mathcal{S} is a set of pp-divisors, we see that, in the case of $\dim Y = 1$ or $Y = \mathbb{P}^n$, the condition from Lemma 6.7 comes down to the ampleness of $\mathcal{D}(u)$ or, alternatively, to $\mathcal{D}(u) \sim 0$ and $Z = \emptyset$. In particular, the sufficient condition from Corollary 6.8 is necessary, too. Thus, to characterize the divisorial fan property, it remains to ask for $\bigcup \{D \mid \Delta_D^i \cap \Delta_D^j = \emptyset\} = \emptyset$ whenever $\mathcal{D}^j(u^{ij}) \sim 0$. \square

7. SEPARATENESS AND COMPLETENESS

Separateness of a toric variety is reflected by the fact that any two cones of its fan admit a separating linear form (cutting out precisely their intersection). Moreover, a toric variety is complete if and only if the cones of its fan cover the whole vector space. In this section, we extend these two observations to the setting of divisorial fans.

The idea is to interpret the valuative criteria for separateness and completeness in our combinatorial terms. We will work with divisorial fans on smooth semiprojective varieties Y ; this is no loss of generality, because, given a divisorial fan \mathcal{S} on a singular Y , we may resolve singularities, then pull back \mathcal{S} , and the new divisorial fan defines the same T -variety.

Let us briefly fix the notation concerning valuations. As usual, we mean by a valuation of $\mathbb{K}(Y)/\mathbb{K}$, where Y is any variety, a valuation $\mu: \mathbb{K}(Y)^* \rightarrow \mathbb{Q}$ of the function field with $\mu = 0$ along \mathbb{K} . Moreover, we say that $y \in Y$ is the (unique) center of μ if the valuation ring $(\mathcal{O}_\mu, \mathfrak{m}_\mu)$ dominates the local ring $(\mathcal{O}_y, \mathfrak{m}_y)$, this means that $\mathcal{O}_y \subseteq \mathcal{O}_\mu$ and $\mathfrak{m}_y = \mathfrak{m}_\mu \cap \mathcal{O}_y$ hold.

Definition 7.1. Let Y be a variety, and let μ be a valuation of $\mathbb{K}(Y)/\mathbb{K}$ with center $y \in Y$. Then there is a well defined group homomorphism

$$\mu: \text{CDiv}(Y) \rightarrow \mathbb{Q}, \quad D \mapsto \mu(f), \quad \text{where } D = \text{div}(f) \text{ near } y \text{ with } f \in \mathbb{K}(Y),$$

and, for smooth Y , this provides a weight function $\mu: \{\text{prime divisors on } Y\} \rightarrow \mathbb{Q}$.

Remark 7.2. The homomorphism $\mu: \text{CDiv}(Y) \rightarrow \mathbb{Q}$ associated to a valuation μ of $\mathbb{K}(Y)/Y$ satisfies $\mu(D) \geq 0$ for all $D \geq 0$.

Recall from Section 6 that, for a divisorial fan \mathcal{S} , we have defined the notion of a slice. For valuations μ , the directed systems $\mu(\mathcal{S})$ of polyhedra in $N_{\mathbb{Q}}$ are weighted versions of our former \mathcal{S}_y . In particular, they share the property of being a complex.

Definition 7.3. Let Y be a smooth semiprojective variety, and let \mathcal{S} be a divisorial fan on Y with polyhedral coefficients living in $N_{\mathbb{Q}}$.

- (i) We say that \mathcal{S} is *separated*, if for any two $\mathcal{D}^i, \mathcal{D}^j \in \mathcal{S}$ and any valuation μ on Y , we have $\mu(\mathcal{D}^i \cap \mathcal{D}^j) = \mu(\mathcal{D}^i) \cap \mu(\mathcal{D}^j)$.

- (ii) We say that \mathcal{S} is *complete*, if Y is complete, \mathcal{S} is separated, and, for every valuation μ , the slice $\mu(\mathcal{S})$ covers $N_{\mathbb{Q}}$.

Remark 7.4. Let \mathcal{S} be a divisorial fan on a smooth semiprojective variety Y .

- (i) The divisorial fan \mathcal{S} is separated if and only if for every valuation μ , the slice $\mu(\mathcal{S})$ is a polyhedral subdivision.
- (ii) If Y is a smooth curve or if \mathcal{S} is coherent, then \mathcal{S} is automatically separated.
- (iii) The divisorial fan \mathcal{S} is complete if and only if Y is complete and, for every valuation μ , the slice $\mu(\mathcal{S})$ is a complete polyhedral subdivision.
- (iv) If Y is a smooth, complete curve, then \mathcal{S} is complete if all prime divisor slices of \mathcal{S} cover the whole vector space.

Theorem 7.5. *Let Y be a smooth semiprojective variety, let \mathcal{S} be a divisorial fan on Y , and let X be the associated prevariety.*

- (i) *X is separated if and only if \mathcal{S} is separated.*
- (ii) *X is complete if and only if \mathcal{S} is complete.*

The proof of this result is based on a characterization of existence of centers for valuations of the function field of an affine T -variety in terms of its defining pp-divisor. A first step is to understand the valuations themselves.

Remark 7.6. Let \mathcal{D} be a pp-divisor with tail cone $\sigma \subseteq N_{\mathbb{Q}}$ on a smooth semiprojective variety Y , and let X be the associated affine T -variety. Then $\mathbb{K}(X)$ is the quotient field of the Laurent polynomial algebra

$$\mathbb{K}(Y)[M] = \bigoplus_{u \in M} \mathbb{K}(Y) \cdot 1^u,$$

where, as usual, $M = \text{Hom}(N, \mathbb{Z})$ is the dual lattice. Given a valuation μ of the function field $\mathbb{K}(Y)$ and a vector $v \in N$, we obtain a map

$$\nu_{\mu, v}: \mathbb{K}(Y)[M] \rightarrow \mathbb{Q}, \quad \sum_i f_i 1^{u_i} \mapsto \min_i (\mu(f_i) + \langle u_i, v \rangle).$$

This map extends to a valuation of the field $\mathbb{K}(X)$, and we have a canonical injection

$$\text{valuations}(\mathbb{K}(Y)/\mathbb{K}) \times N \rightarrow \text{valuations}(\mathbb{K}(X)/\mathbb{K}), \quad (\mu, v) \mapsto \nu_{\mu, v}.$$

Conversely, any valuation ν on $\mathbb{K}(X)/\mathbb{K}$ coincides on the homogeneous elements of $\mathbb{K}(Y)[M]$ with a unique $\nu_{\mu, v}$: the data μ and v are defined via

$$\mu = \nu|_{\mathbb{K}(Y)}, \quad \langle u, v \rangle = \nu(1^u).$$

Thus, on the homogeneous elements of $\mathbb{K}(Y)[M]$, any valuation ν of $\mathbb{K}(X)$ is uniquely represented by a valuation $\nu_{\mu, v}$; we will denote this by $\nu \sim \nu_{\mu, v}$.

Lemma 7.7. *Let \mathcal{D} be a pp-divisor on a smooth semiprojective variety Y , let X be the associated affine T -variety, and consider a valuation $\nu \sim \nu_{\mu, v}$ on $\mathbb{K}(X)/\mathbb{K}$. Then the following statements are equivalent.*

- (i) *The valuation ν on $\mathbb{K}(X)/\mathbb{K}$ has a center $x \in X$.*
- (ii) *The valuation μ on $\mathbb{K}(Y)/\mathbb{K}$ has a center $y \in Y$ with $v \in \mu(\mathcal{D})$.*

Proof. As usual, let $A = \Gamma(Y, \mathcal{A}) = \Gamma(X, \mathcal{O})$ denote the global ring associated to the pp-divisor \mathcal{D} .

Suppose that ν has a center $x \in X$. Then we obtain $A \subseteq \mathcal{O}_\nu$, which implies $A_0 \subseteq \mathcal{O}_\nu$. Thus, there is a center in $Y_0 := \text{Spec}(A_0)$ for the restriction of ν to $\mathbb{K}(Y_0)$. Since μ and ν coincide on $\mathbb{K}(Y_0)$ and Y is projective over Y_0 , the valuative criterion of properness [Har77, Thm II.4.7] provides a center $y \in Y$ for μ .

In order to verify the desired property for the weight function μ , suppose, to the contrary, that $v \notin \mu(\mathcal{D})$ holds. Then, there is a linear form $u \in \sigma^\vee \cap M$, where σ stands for the tail cone of \mathcal{D} , such that

$$\langle u, v \rangle < \min \langle u, \mu(\mathcal{D}) \rangle = \mu(\mathcal{D}(u)).$$

Replacing u with a suitable positive multiple, we achieve that $\mathcal{D}(u)$ is Cartier and base point free on Y . Then we may choose a global section $f \in \Gamma(Y, \mathcal{D}(u))$ such that f^{-1} is a local equation for $\mathcal{D}(u)$ near y . We obtain

$$\mu(f) = -\mu(\mathcal{D}(u)) < -\langle u, v \rangle.$$

This implies $\nu_{\mu,v}(f) < 0$, and thus $f \in \Gamma(Y, \mathcal{D}(u)) \subseteq \Gamma(X, \mathcal{O})$ does not belong to the valuation ring \mathcal{O}_ν . Consequently, \mathcal{O}_ν cannot dominate any local ring \mathcal{O}_x , meaning that $\nu \sim \nu_{\mu,v}$ has no center in X ; a contradiction.

Now suppose that μ has a center $y \in Y$ and that $\mu: \text{CDiv}(Y) \rightarrow \mathbb{Q}$ is as in (ii). Then $\mu(\mathcal{D}(u)) = \min \langle u, \mu(\mathcal{D}) \rangle \leq \langle u, v \rangle$ holds for all $u \in \sigma^\vee \cap M$. Thus, we have

$$\begin{aligned} f \in \Gamma(Y, \mathcal{O}(\mathcal{D}(u))) &\implies \text{div}(f) \geq -\mathcal{D}(u) \\ &\implies \mu(f) \geq \mu(-\mathcal{D}(u)) \geq \langle u, v \rangle \\ &\implies \nu(f) = \mu(f) + \langle u, v \rangle \geq 0 \\ &\implies f \in \mathcal{O}_\nu. \end{aligned}$$

This implies $A \subseteq \mathcal{O}_\nu$. Thus, there is a prime ideal $\mathfrak{p} \subseteq A$ such that $A_{\mathfrak{p}}$ is dominated by \mathcal{O}_ν . In other words: ν has a center $x \in X$. \square

Proof of Proposition 7.5. We first treat separateness by applying the well-known valuative criterion, cf. [Har77, Thm. II.4.3]. It says that a prevariety X is separated if and only if every valuation of $\mathbb{K}(X)/\mathbb{K}$ admits at most one center in X .

Let \mathcal{S} be separated. Consider a valuation $\nu \sim \nu_{\mu,v}$ of $\mathbb{K}(X)/\mathbb{K}$ with centers $x, x' \in X$. Then x and x' belong to affine charts $X(\mathcal{D})$ and $X(\mathcal{D}')$ with $\mathcal{D}, \mathcal{D}' \in \mathcal{S}$. By Lemma 7.7, there is a (unique) center $y \in Y$ for μ , and we have

$$v \in \mu(\mathcal{D}) \cap \mu(\mathcal{D}').$$

Separateness of \mathcal{S} gives $v \in \mu(\mathcal{D} \cap \mathcal{D}')$. Again by Lemma 7.7, it follows that ν has a center in $X(\mathcal{D} \cap \mathcal{D}') = X(\mathcal{D}) \cap X(\mathcal{D}')$. Since $X(\mathcal{D})$ and $X(\mathcal{D}')$ are separated, this implies $x = x'$.

Now, let X be separated, and suppose that \mathcal{S} is not. Then there are $\mathcal{D}, \mathcal{D}' \in \mathcal{S}$ and a valuation μ on Y such that we have

$$\mu(\mathcal{D} \cap \mathcal{D}') \subsetneq \mu(\mathcal{D}) \cap \mu(\mathcal{D}').$$

Pick any $v \in \mu(\mathcal{D}) \cap \mu(\mathcal{D}')$ that does not belong to $\mu(\mathcal{D} \cap \mathcal{D}')$. Then by Lemma 7.7, the valuation $\nu = \nu_{\mu,v}$ has no center in $X(\mathcal{D} \cap \mathcal{D}')$ but it has centers $x \in X(\mathcal{D}')$ and $x' \in X(\mathcal{D})$. Since $X(\mathcal{D} \cap \mathcal{D}')$ equals $X(\mathcal{D}) \cap X(\mathcal{D}')$, we obtain $x \neq x'$. This contradicts separateness of X .

We turn to completeness. Again we make use of a valuative criterion, cf. [Har77, Thm. II.4.7]. It says that a (separated) variety X is complete if and only if every valuation of $\mathbb{K}(X)/\mathbb{K}$ has a center in X .

Let X be complete. Suppose that Y is not. Then there is a valuation μ of $\mathbb{K}(Y)/\mathbb{K}$ without center on Y . Take any $v \in N_{\mathbb{Q}}$. Lemma 7.7 says that $\nu_{\mu,v}$ has no center in X . A contradiction. Next suppose that there were a noncomplete valuative slice $\mu(\mathcal{S})$. Take $v \in N_{\mathbb{Q}} \setminus |\mu(\mathcal{S})|$. Then, again by Lemma 7.7, the valuation $\nu_{\mu,v}$ on $\mathbb{K}(X)/\mathbb{K}$ has no center in X . A contradiction.

Finally, let \mathcal{S} be complete. Let $\nu \sim \nu_{\mu,v}$ be any valuation on $\mathbb{K}(X)/\mathbb{K}$. By completeness of Y , the valuation μ has a center $y \in Y$. Moreover, the slice $\mu(\mathcal{S})$ is complete, this means that there is a $\mathcal{D} \in \mathcal{S}$ such that $v \in \mu(\mathcal{D})$ holds. Lemma 7.7 then provides a center of ν in $X(\mathcal{D}) \subseteq X$. \square

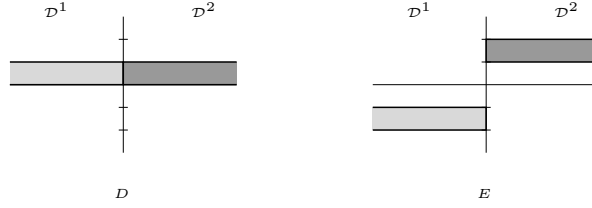
Remark 7.8. Example 6.3 with, e.g., $Y = \mathbb{A}^1$ and D_1, D_2 being two different prime divisors, yields a separated, but non-coherent divisorial fan.

The following two examples underline that for checking separateness and completeness it is not sufficient to consider the prime divisor slices.

Example 7.9. Let $Y := \mathbb{A}^2$ with the coordinate functions z and w , and denote the coordinate axes by $D := \text{div}(z)$ and $E := \text{div}(w)$. We consider pp-divisors

$$\mathcal{D}^1 = \Delta_D^1 \otimes D + \Delta_E^1 \otimes E, \quad \mathcal{D}^2 = \Delta_D^2 \otimes D + \Delta_E^2 \otimes E$$

with tail cones $\text{tail}(\mathcal{D}^1) = \mathbb{Q}_{\leq 0} \cdot e_1$ and $\text{tail}(\mathcal{D}^2) = \mathbb{Q}_{\geq 0} \cdot e_1$ and polyhedral coefficients in \mathbb{Q}^2 according to the following figure:



Then $\mathcal{S} := \{\mathcal{D}^1, \mathcal{D}^2\}$ generates a divisorial fan, and the above figure shows that its prime divisor slices are nice polyhedral subdivisions. However, the divisorial fan \mathcal{S} is not separated. While the weight function $\mu_{(0,0)}$ yields the subdivision $\mathcal{S}_{(0,0)}$ consisting of the polytopes $\mathcal{D}_{(0,0)}^i = \Delta_D^i + \Delta_E^i$ and \emptyset , the valuation

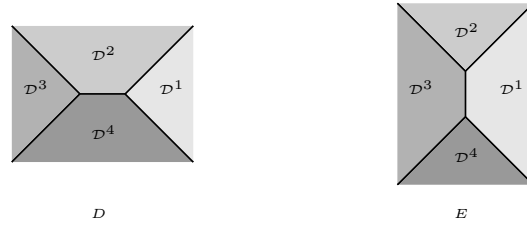
$$\mu(\sum_{a,b} \lambda_{a,b} z^a w^b) := \min\{2a + b \mid \lambda_{a,b} \neq 0\}$$

provides $\mu(\mathcal{S}) = \{\mu(\mathcal{D}^1), \mu(\mathcal{D}^2), \mu(\mathcal{D}^1 \cap \mathcal{D}^2)\}$ with

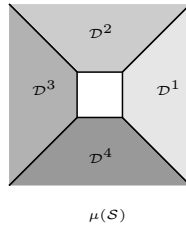
$$\mu(\mathcal{D}^1 \cap \mathcal{D}^2) = \emptyset, \text{ but } \mu(\mathcal{D}^1) \cap \mu(\mathcal{D}^2) = \{(0,1)\}.$$

One also sees directly that $X(\mathcal{S})$, as the gluing of $\text{Spec}(\mathbb{K}[x, y, s, ty^{-1}, xy^2t^{-1}])$ and $\text{Spec}(\mathbb{K}[x, y, s^{-1}, ty^2, xt^{-1}y^{-1}])$ along $\text{Spec}(\mathbb{K}[x, y, y^{-1}, s, s^{-1}, t, xt^{-1}])$, is not separated.

Example 7.10. On $Y := \mathbb{P}^2$, fix two coordinate axes, say D and E , and consider the coherent set $\mathcal{S} = \{\mathcal{D}^1, \dots, \mathcal{D}^4\}$ of polyhedral divisors given by its prime divisor slices as indicated below. By Proposition 6.9, we know that $\langle \mathcal{S} \rangle$ is a divisorial fan.



The prime divisor slices are complete, but there is a valuation μ with $\mu(D) = \mu(E) = 1$ providing the following noncomplete slice.

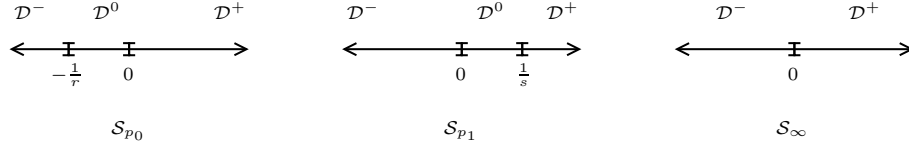


8. FURTHER EXAMPLES

Example 8.1. In [FKZ05, Example 5.13], Flenner et al. describe the so-called Danilov-Gizatullin \mathbb{K}^* -surfaces V_0 . They depend on the choice of $r, s \in \mathbb{Z}_{\geq 1}$ and points $p_0, p_1 \in \mathbb{K}^1$, and, using our language of pp-divisors, are given by

$$\mathcal{D}^0 = [-\frac{1}{r}, 0] \otimes \{p_0\} + [0, \frac{1}{s}] \otimes \{p_1\} \in \text{PPDiv}_{\mathbb{Q}}(\mathbb{K}^1, \{0\})$$

with $N = \mathbb{Z}$. In [FKZ05], the surface V_0 is described by the two \mathbb{Q} -divisors $D_+ = \mathcal{D}^0(1) = -\frac{1}{r}p_0$ and $D_- = \mathcal{D}^0(-1) = -\frac{1}{s}p_1$ on \mathbb{K}^1 . Proposition 3.8 of [FKZ05] provides a canonical \mathbb{K}^* -equivariant completion of V_0 by “adding” two charts V_+ and V_- . Using our divisorial fans, this completion is given as $\tilde{X}(\mathcal{S})$ with $\mathcal{S} = \{\mathcal{D}^-, \mathcal{D}^0, \mathcal{D}^+\}$



(and $\mathcal{D}_{\infty}^0 = \emptyset$) on $\mathbb{P}_{\mathbb{K}}^1$. Note that the contraction $X(\mathcal{S})$ of $\tilde{X}(\mathcal{S})$ provides an even “smaller” compactification of V_0 . The completeness of $X(\mathcal{S})$ and $\tilde{X}(\mathcal{S})$ is reflected by the fact that, for every prime divisor, $M = \mathbb{Q}$ is completely covered by the corresponding polyhedral coefficients (cf. Theorem 7.5).

Example 8.2. In [KKMS, IV.1] Mumford constructed a toroidal structure on normal varieties X with one-codimensional torus action. This leads to an associated fan carrying some information about the original X . However, in contrast to the case of toric varieties, this fan is not sufficient to recover X .

Since the theory of divisorial fans developed in the present paper provides a tool for keeping complete information about the normal varieties with their torus action, there should be a direct way to translate divisorial fans on curves Y into the Mumford fans. For affine X , this has been done by Vollmert in [Vo07]: The Mumford fan is obtained by gluing the homogenizations of the polyhedral coefficients of the pp-divisor along the tail cone that is their common face. As Mumford’s construction works for non-affine varieties, too, one easily sees that Vollmert’s theorem remains valid in the general case.

In the next examples, we consider equivariant vector bundles on a given toric variety $X = \text{TV}(\Sigma)$ arising from a fan Σ in $N_{\mathbb{Q}}$.

Recall that Klyachko [Kly90], and later Perling [Per04], gave a combinatorial description of the T -equivariant, reflexive sheaves \mathcal{E} on X . In Perling’s notation, \mathcal{E} is given by a \mathbb{K} -vector space E together with \mathbb{Z} -labeled increasing filtrations $E^{\varrho}(i)$ for every $\varrho \in \Sigma^{(1)}$. Note that not only is the filtrations itself an important data, but also the i , which tells you when a jump takes place.

The sheaf \mathcal{E} is locally free (of rank $r = \dim_{\mathbb{K}} E$) if and only if we can find for every $\sigma \in \Sigma$ a basis $e_1^{\sigma} \dots e_r^{\sigma}$ of E and weights $u_1^{\sigma}, \dots, u_r^{\sigma} \in M$ such that for all $\varrho \in \sigma(1)$ one has

$$e_j^{\sigma} \in E^{\varrho}(i) \iff \langle u_j^{\sigma}, \varrho \rangle \geq i.$$

For the affine charts $\text{TV}(\sigma) \subseteq X$, we have $\mathcal{E}(\text{TV}(\sigma)) \subseteq \mathcal{E}(T) \cong E \otimes_{\mathbb{K}} \mathbb{K}[M]$; thus these data corresponds directly to elements of the graded module $\mathcal{E}(\text{TV}(\sigma))$ which generate $\mathcal{E}|_{\text{TV}(\sigma)}$ freely.

There are two striking examples for this encoding via filtrations. First, reflexive sheaves of rank one utilize a one-dimensional vector space E . Since their filtrations are completely determined by telling for which i_{ϱ} the unique step

$$E^{\varrho}(i_{\varrho} - 1) = 0, \quad E^{\varrho}(i_{\varrho}) = E$$

takes place, Klyachko's description just means to fix a map $i : \Sigma^{(1)} \rightarrow \mathbb{Z}$. This coincides with the classical description, see, e.g., [KKMS]. Second, the cotangent bundle Ω_X on a smooth $X = \mathbb{T}\mathbb{V}(\Sigma)$ may be obtained from the vector space $E := M \otimes_{\mathbb{Z}} \mathbb{K}$ with the filtrations

$$E^e(0) = E, \quad E^e(1) = \mathbb{K} \cdot \varrho^\perp, \quad E^e(2) = E.$$

If $\mathcal{E} = \oplus_j \mathcal{L}_j$ is a splitting vector bundle on $X = \mathbb{T}\mathbb{V}(\Sigma)$, then $\mathbb{P}(\mathcal{E})$ is toric again (under a larger torus). Assume, e.g., that the direct summands \mathcal{L}_j of \mathcal{E} are ample and given by lattice polytopes $\Delta_j \subseteq M_{\mathbb{Q}}$. Then, $\mathbb{P}(\mathcal{E})$ is associated to the normal fan of

$$\tilde{\Delta} := \text{conv}\left(\bigcup_j \Delta_j \times \{e_j\}\right) \subseteq M_{\mathbb{Q}} \times \mathbb{Q}^m.$$

If \mathcal{E} does not split, then $\mathbb{P}(\mathcal{E})$ admits a torus action of our original, lower-dimensional T . This suggests understanding $\mathbb{P}(\mathcal{E})$ as a divisorial fan \mathcal{S} on some variety Y . The slices of \mathcal{S} should be polyhedral subdivisions of $N_{\mathbb{Q}}$.

Example 8.3. Consider an equivariant rank 2 bundle \mathcal{E} on a toric variety, given by some vector space filtrations $E^e(i)$ of $E = \mathbb{K}^2$. Set $Y = \mathbb{P}^1 \cong \mathbb{P}(E^*)$. For every maximal cone σ , we obtain coefficients for two polyhedral divisors by cutting σ with affine hyperplanes orthogonal to $u_1^\sigma - u_2^\sigma$.

$$\Delta_\sigma^1 = \{v \in N_{\mathbb{Q}} \mid \langle u_1^\sigma - u_2^\sigma, v \rangle \geq 1\} \cap \sigma, \quad \Delta_\sigma^2 = \{v \in N_{\mathbb{Q}} \mid \langle u_2^\sigma - u_1^\sigma, v \rangle \geq 1\} \cap \sigma$$

$$\nabla_\sigma^1 = \{v \in N_{\mathbb{Q}} \mid \langle u_1^\sigma - u_2^\sigma, v \rangle \leq 1\} \cap \sigma, \quad \nabla_\sigma^2 = \{v \in N_{\mathbb{Q}} \mid \langle u_2^\sigma - u_1^\sigma, v \rangle \leq 1\} \cap \sigma$$

The participating prime divisors for the σ -chart are $\{(e_1^\sigma)^\perp\}, \{(e_2^\sigma)^\perp\} \in \mathbb{P}^1$, where $(e_i^\sigma)^\perp$ is the one dimensional subspace of E^* orthogonal to e_i and thus a element of Y . To be precise, we define the polyhedral divisors $\mathcal{D}_\sigma^+, \mathcal{D}_\sigma^-$ on $Y = \mathbb{P}^1$.

$$\begin{aligned} \mathcal{D}_\sigma^+ &= \Delta_\sigma^1 \otimes \{(e_1^\sigma)^\perp\} + \Delta_\sigma^2 \otimes \{(e_2^\sigma)^\perp\} \\ \mathcal{D}_\sigma^- &= \nabla_\sigma^1 \otimes \{(e_1^\sigma)^\perp\} + \nabla_\sigma^2 \otimes \{(e_2^\sigma)^\perp\}. \end{aligned}$$

Proposition 8.4. *The set $\{\mathcal{D}_\sigma^\pm \mid \sigma \in \Sigma^{\max}\}$ of all these polyhedral divisors generates a fan which encodes $\mathbb{P}(\mathcal{E})$.*

Proof. We consider a maximal cone $\sigma \in \Sigma^{\max}$. For now we fix the coordinates of \mathbb{P}^1 given by the dual basis $(e_1^\sigma)^*, (e_2^\sigma)^*$. Then we have

$$\mathcal{D}_\sigma^+ = \Delta_\sigma^1 \otimes \{0\} + \Delta_\sigma^2 \otimes \{\infty\}.$$

We consider $\mathbb{K}[M][\frac{x_1}{x_2}, \frac{x_2}{x_1}]$ with the obvious M -grading. Then the elements with weight u are exactly those of the form $\chi^u f$ with $f \in \mathbb{K}[\frac{x_1}{x_2}, \frac{x_2}{x_1}] \subset \mathbb{K}(\mathbb{P}^1)$. With this representation, we have the graded isomorphisms

$$\begin{aligned} \mathbb{K}[\sigma^\vee \cap M][\chi^{u_1^\sigma - u_2^\sigma} \frac{x_1}{x_2}] &\rightarrow \bigoplus_u \Gamma(\mathcal{O}(\mathcal{D}_\sigma^+(u))) \\ \chi^u f &\mapsto f \in \Gamma(\mathcal{O}(\mathcal{D}_\sigma^+(u))) \end{aligned}$$

$$\begin{aligned} \mathbb{K}[\sigma^\vee \cap M][\chi^{u_2^\sigma - u_1^\sigma} \frac{x_2}{x_1}] &\rightarrow \bigoplus_u \Gamma(\mathcal{O}(\mathcal{D}_\sigma^-(u))) \\ \chi^u f &\mapsto f \in \Gamma(\mathcal{O}(\mathcal{D}_\sigma^-(u))) \end{aligned}$$

$$\begin{aligned} \mathbb{K}[\sigma^\vee \cap M][\chi^{u_1^\sigma - u_2^\sigma} \frac{x_1}{x_2}, \chi^{u_2^\sigma - u_1^\sigma} \frac{x_2}{x_1}] &\rightarrow \bigoplus_u \Gamma(\mathcal{O}((\mathcal{D}_\sigma^+ \cap \mathcal{D}_\sigma^-)(u))) \\ \chi^u f &\mapsto f \in \Gamma(\mathcal{O}((\mathcal{D}_\sigma^+ \cap \mathcal{D}_\sigma^-)(u))) \end{aligned}$$

Thus, we obtain $\mathbb{P}(\mathcal{E}|_{X_\sigma})$ by gluing $X(\mathcal{D}_\sigma^+)$ and $X(\mathcal{D}_\sigma^-)$. For getting the global result we also need to take into account the base change $e_1^\sigma, e_2^\sigma \mapsto e_1^\delta, e_2^\delta$ between two cones. \square

Example 8.5 (Cotangent bundle). Let $X = \mathbb{T}\mathbb{V}(\Sigma)$ be a smooth toric variety. For every (maximal) cone $\sigma \in \Sigma$ we consider the polyhedra:

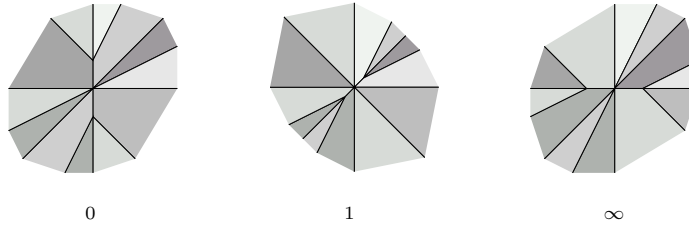
$$\Delta_\sigma^{ij} = \bigcap_{k=1}^n \{v \in N_{\mathbb{Q}} \mid \langle u_k^\sigma - u_i^\sigma, v \rangle \geq \delta_{ij} - \delta_{jk}\} \cap \sigma,$$

where $u_1^\sigma, \dots, u_n^\sigma$ is a \mathbb{Z} -basis of M , such that the dual basis $(u_1^\sigma)^* \dots (u_n^\sigma)^*$ contains the primitive generators of the rays in $\sigma(1)$. We set $Y = \mathbb{P}(N \otimes \mathbb{K})$ and

$$\mathcal{D}_\sigma^i = \sum_{j=1}^n \Delta_\sigma^{ij} \otimes (u_j^\sigma)^\perp.$$

Then the divisorial fan $\mathcal{S}_\Omega(\Sigma)$ generated by the set $\{\mathcal{D}_\sigma^i\}_{\sigma,i}$ describes the cotangent bundle on X . Note that the tail fan of $\mathcal{S}_\Omega(\Sigma)$ is obtained by a barycentric subdivision of the n -dimensional cones of Σ .

Here are two concrete examples. Firstly, for $\Omega_{\mathbb{P}^2}$ on \mathbb{P}^2 we obtain the picture already shown in the introduction. Secondly, for the cotangent bundle Ω_{dP_6} on the del Pezzo surface dP_6 , we obtain the divisorial fan given by following prime divisor slices.



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FACHBEREICH MATHEMATIK UND INFORMATIK, FREIE UNIVERSITÄT BERLIN, ARNIMALLE 3,
14195 BERLIN, GERMANY

E-mail address: `altmann@math.fu-berlin.de`

MATHEMATISCHES INSTITUT, UNIVERSITÄT TÜBINGEN, AUF DER MORGENSTELLE 10, 72076
TÜBINGEN, GERMANY

E-mail address: `hausen@mail.mathematik.uni-tuebingen.de`

INSTITUT FÜR MATHEMATIK, LS ALGEBRA UND GEOMETRIE, BRANDENBURGISCHE TECHNISCHE
UNIVERSITÄT COTTBUS, PF 10 13 44, 03013 COTTBUS, GERMANY

E-mail address: `suess@math.tu-cottbus.de`